Basics of Fluid Mechanics

with Applications to the Problems of Capillarity

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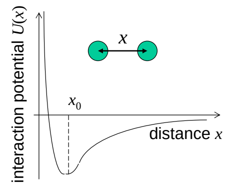
Liquids



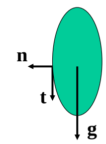
Phase state	Amplitude of thermal motion	Order
Gas	>> x_0	no
Liquid	~ x_0	some
Solid	<< x_0	perfect

Body forces and surface forces

Consider a body suspended in a liquid

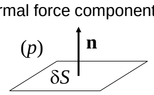
$$\mathbf{F}_{\text{gravity}} = \mathbf{g}\iiint_V \rho \, dV$$ (*body force*)

$$\mathbf{F}_{\text{buoyancy}} = -\oiint_S p\mathbf{n}\, dS$$ (*surface force*)



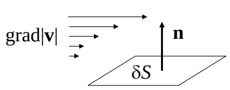
Representation of surface forces by stress tensor

Normal force component:



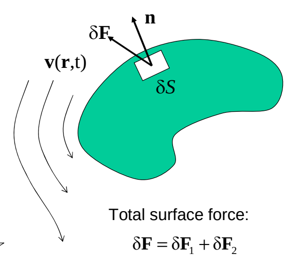
$$\delta\mathbf{F}_1 = -p\mathbf{n}\delta S$$

Tangential force component:



$$\delta\mathbf{F}_2 = \frac{\mathbf{v}}{|\mathbf{v}|}\eta\frac{\partial|\mathbf{v}|}{\partial\mathbf{n}}\delta S$$

$[\eta]$ = Pa·s



Total surface force:

$$\delta\mathbf{F} = \delta\mathbf{F}_1 + \delta\mathbf{F}_2$$

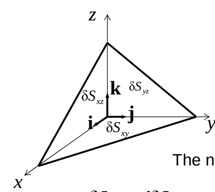
has a direction different from that of the normal.

$$\delta\mathbf{F} = \underbrace{\boldsymbol{\sigma}}_{\text{local stress}} \cdot \mathbf{n}$$

tensor x vector = vector

Why the stress Π is a tensor?

Surface force depends upon the direction of the normal to the surface across which it acts. Let's analyze this dependence:



$$\underbrace{\iiint_V \rho(\mathbf{r})[\mathbf{f}(\mathbf{r}) - \mathbf{r}'']dV}_{\text{body force}} + \underbrace{\oiint_S \boldsymbol{\sigma}\cdot\mathbf{n}\,dS}_{\text{surface force}} = 0$$

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1; \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{i}\cdot\mathbf{k} = 0$$

The net surface force exerted on the tetrahedron:

$$\boldsymbol{\sigma}\cdot\mathbf{n}\delta S - \boldsymbol{\sigma}\cdot\mathbf{i}\delta S_{yz} - \boldsymbol{\sigma}\cdot\mathbf{j}\delta S_{xz} - \boldsymbol{\sigma}\cdot\mathbf{k}\delta S_{xy}$$

$$= \{\boldsymbol{\sigma}\cdot\mathbf{n} - [\boldsymbol{\sigma}\cdot\mathbf{i}\underbrace{(\mathbf{i}\cdot\mathbf{n})}_{n_x} + \boldsymbol{\sigma}\cdot\mathbf{j}\underbrace{(\mathbf{j}\cdot\mathbf{n})}_{n_y} + \boldsymbol{\sigma}\cdot\mathbf{k}\underbrace{(\mathbf{k}\cdot\mathbf{n})}_{n_z}]\}\delta S \to 0 \quad \text{as} \quad \delta V \to 0$$

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \boldsymbol{\sigma}\cdot\mathbf{i}n_x + \boldsymbol{\sigma}\cdot\mathbf{j}n_y + \boldsymbol{\sigma}\cdot\mathbf{k}n_z$$

Why is the stress Π is a tensor? (cont'd)

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \boldsymbol{\sigma}\cdot\mathbf{i}n_x + \boldsymbol{\sigma}\cdot\mathbf{j}n_y + \boldsymbol{\sigma}\cdot\mathbf{k}n_z$$

$$\Updownarrow$$

$$\begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{pmatrix}\begin{pmatrix}n_x\\ n_y\\ n_z\end{pmatrix} = \begin{pmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{pmatrix}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}n_x + \begin{pmatrix}0\\1\\0\end{pmatrix}n_y + \begin{pmatrix}0\\0\\1\end{pmatrix}n_z\right\}$$

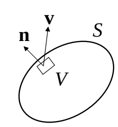
Thus, Π meets the transformation properties for a 2-nd rank tensor.

Basic equations of fluid mechanics

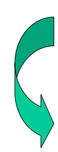
Continuity constraint:

$$\underbrace{\frac{\partial}{\partial t}\int \rho\, dV}_{\text{change in the mass of volume } V} = -\underbrace{\oint \rho\mathbf{v}\cdot d\mathbf{S}}_{\text{amount of substance entering or leaving the volume per unit time}}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})$$



Euler equation:



$$\underbrace{\frac{d}{dt}\int \rho\mathbf{v}\,dV = F}_{\text{the Newton equation applied to a fixed mass of fluid}} = \underbrace{-\oint p\,d\mathbf{S}}_{\text{surface force acting on the volume } V} = -\int \nabla p\, dV$$

$$\underbrace{\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}}_{\text{Lagrangian derivative } d\mathbf{v}/dt} = -\frac{1}{\rho}\nabla p$$

dynamics of inviscid fluids (no dissipation of energy)

Liquid in the field of gravity

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

An alternative form of the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$\nabla\times$

rot or curl operation

$$\nabla\times\mathbf{a} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{pmatrix}$$

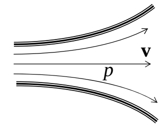
$$\frac{\partial}{\partial t}\nabla\times\mathbf{v} + \nabla\times[\mathbf{v}\times[\nabla\times\mathbf{v}]] = 0$$

RHS is zero because $\nabla\times(\nabla p)\equiv 0.$

Boundary condition:
liquid can't go through the wall

$$\mathbf{v}\cdot\mathbf{n}|_S = 0$$

Stationary flow: Bernoulli's equation



zero because velocity doesn't change with time

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$$\tfrac{1}{2}\nabla(\mathbf{v}\cdot\mathbf{v}) = [\mathbf{v}\times[\nabla\times\mathbf{v}]] + (\mathbf{v}\cdot\nabla)\mathbf{v}$$

$$\nabla\left(\tfrac{1}{2}\rho v^2 + p\right) = \underbrace{[\mathbf{v}\times[\nabla\times\mathbf{v}]}_{\text{perpendicular to flow lines}}$$

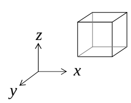
Along the flow lines:

$$\tfrac{1}{2}\rho v^2 + p = \text{const}$$

principle of water pump operation

Momentum flux density

$$\frac{\partial}{\partial t}(\rho v_i) = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t}$$



$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{\rho}\nabla p$ *(Euler equation)*

$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})$ *(continuity)*

$$\frac{\partial}{\partial t}(\rho v_i) = -\rho v_k\frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i\frac{\partial}{\partial x_k}(\rho v_k)$$

$$= -\frac{\partial}{\partial x_k}\underbrace{(p\delta_{ik} + \rho v_i v_k)}_{\substack{\text{momentum flux}\\ \text{density tensor}\\ (\Pi_{ik})}}$$

time-derivative of momentum

force

Implicit summation! (Einstein's convention)

Dynamics of a viscous fluid: Navier-Stokes equation

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \mathbf{\Pi}_{ik}}{\partial x_k}$$

For a viscous fluid there is an additional "viscous" term in Π and σ,

momentum flux

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \underbrace{\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)}_{\text{viscous stress}}$$

stress

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

η - dynamic viscosity

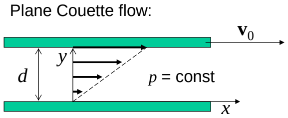
The corresponding dynamic equation (Navier-Stokes) is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \underbrace{\frac{\eta}{\rho}\Delta\mathbf{v}}_{\text{due to viscosity}}$$

Applications of Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\Delta\mathbf{v}$$

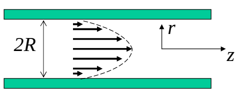
Plane Couette flow:



$$0 = \eta\frac{\partial^2 v_x}{\partial y^2}$$

$$v_x(y) = \frac{v_0 y}{d}$$

Flow in a cylindrical tube:

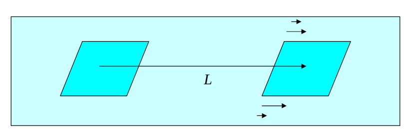


$$0 = -\underbrace{\frac{\partial p}{\partial z}}_{K=\text{const}} + \frac{\eta}{\rho}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$v_z(r) = \frac{KR^2}{4\eta}\left(1 - \frac{r^2}{R^2}\right)$$

$$Q = \int_0^R 2\pi r v_z(r)\,\mathrm{d}r = \frac{\pi K R^4}{8\eta}$$

Reynolds number



Convection-related momentum flux

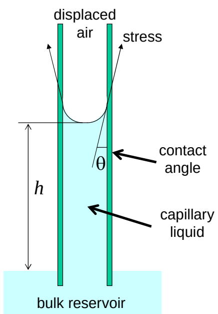
$$\frac{\text{momentum}}{\text{time}\times\text{area}} \sim \frac{\rho L^3 v}{\frac{L}{v}\times L^2} \sim \rho v^2 \;\; [\mathrm{N\cdot m^{-2}}]$$

Viscosity-related momentum flux

$$\frac{\overbrace{\text{viscous force}\times\text{time}}^{\text{momentum}}}{\text{area}\times\text{time}} \sim \frac{\eta L^2\frac{v}{L}\times\frac{L}{v}}{L^2\times\frac{L}{v}} \sim \frac{\eta v}{L} \;\; [\mathrm{N\cdot m^{-2}}]$$

$$Re = \frac{\rho v L}{\eta}$$

Capillary rise: Familiar and Unfamiliar



Hydrostatic equilibrium:

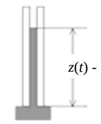
$$\underbrace{2\pi r}_{\text{perimeter of meniscus}}\ \gamma\cos\theta = \underbrace{\pi r^2 h}_{\text{volume of capillary liquid}}\ \rho g$$

γ - surface tension
ρ - density
g - acceleration of gravity

Equilibrium rise height:

$$h = \frac{2\gamma\cos\theta}{r\rho g}$$

The dynamics of capillary rise



Full dynamic equation (*Newton's law*):

$$\frac{\mathrm{d}p}{\mathrm{d}t} = F$$

p – momentum $(p = mv = mz' = \pi r^2 \rho z z')$

$$\underbrace{\pi r^2 \rho[zz'' + (z')^2]}_{\text{time-derivative of momentum}} = \underbrace{2\pi r\gamma\cos\theta}_{\text{capillary force}} - \underbrace{8\pi\eta z z'}_{\text{viscous drag (Poiseuille flow)}} - \underbrace{\pi r^2 z\rho g}_{\text{gravity force}}$$

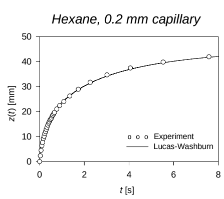
Lucas-Washburn approximation (*quasi-steady rise, low Re*):

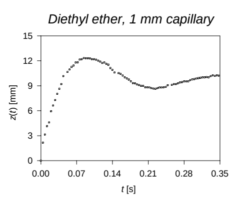
$$0 = \frac{2}{r}\gamma\cos\theta - \frac{8}{r^2}\eta z z' - \rho g z$$

$$z' = \frac{r^2}{8\eta z}\left(\frac{2\gamma\cos\theta}{r} - \rho g z\right) \Rightarrow t = \frac{8\eta}{r^2\rho g}\left(z_\infty \ln\frac{z_\infty}{z_\infty - z(t)} - z(t)\right)$$

In the short-time limit: $z(t) = \sqrt{\frac{r\gamma\cos\theta}{2\eta}t} \quad (t \to 0)$

Capillary rise dynamics in real systems





Quasi-steady rise, LW equation works fine

Inertia effects are important, LW equation fails completely

Obvious shortcomings of the above treatment

- Shape of the meniscus is assumed to be fixed
- Infinite acceleration at zero time (touch-point)
- Infinite stress on the three phase contact line

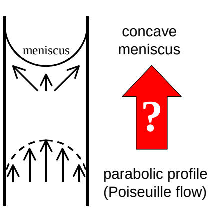
More recent theoretical developments

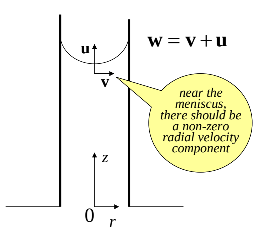
Szekely (1971), Levine (1976) – remedied the "infinite acceleration" fault by revising the energy balance.

van Dyke (1964), van Dussan (1976), Levine (1979) – remedied the "infinite stress" fault by allowing for the boundary layer slippage.

Capillary rise dynamics revisited

(after Levine et al, JCIS, 73 (1980) 136)





Flow continuity: $\mathrm{div}\,\mathbf{w} = 0$

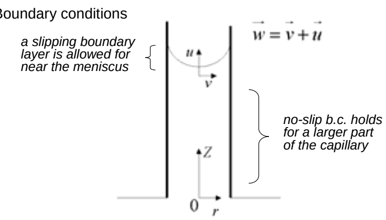
$$\frac{\partial}{\partial r}(rv) + r\frac{\partial u}{\partial z} = 0$$

Navier-Stokes equation for capillary flow

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\eta}{\rho}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right)$$

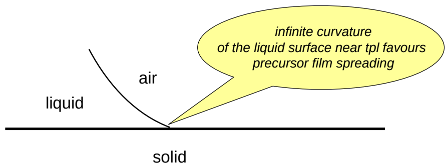
$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\eta}{\rho}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) - g$$

Boundary conditions



Remaining limitations

- Only the quasi-steady rise regime is analyzed
- Meniscus dynamics is neglected
- Entry-point effects are neglected
- Conditions for no-slip to slippage transition and transition zone effects are unknown



Note: Precursor films spreading is driven by a difference in the chemical potential of liquid molecules (i.e. a non-hydrodynamic force)